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FORMULATION OF SOLUTIONS FOR LINEAR DIFFERENTIAL EQUATIONS WITH QUASIPERIODIC COEFFICIENTS BY MEANS OF THE ACCELERATED CONVERGENCE METHOD

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ABSTRACT

The system of n differential equations $\frac{dx}{dt}=[A+P(\omega t)]x$ where A is a constant matrix, and $P(\omega t)$ is a matrix having n basic frequencies is investigated. For the solution, a correction matrix, ξ, is introduced and the solution is obtained in the form $x=\Phi(\omega t)e^{At}x_0$ where Φ is a periodic matrix of period 2π.

1. Let us investigate a system of differential equations /42*

$$\frac{dx}{dt} = [A + P(\omega t)]\,x, \tag{1.1}$$

where A is constant, $P(\omega t)$ -- the quasiperiodic n-dimensional matrix, $\omega = (\omega_1, \omega_2, \ldots, \omega_m)$ -- the basic frequencies of the matrix $P(\omega t)$, $x = (x_1, \ldots, x_n)$ -- n-dimensional vector, t -- time. In addition, let us assume that the matrix $P(\omega t)$ is small.

As is known, N. A. Artemyeva (Ref. 1), N. P. Yerugina (Ref. 9), I. Z. Shtokalo (Ref. 12), A. N. Kolmogorova (Ref. 10), E. G. Begagi (Ref. 5), A. Ye. Gel'mana (Ref. 7-8), L. Ya. Adrianovoy (Ref. 3), Yu. A. Mitropol'skogo (Ref. 11), I. N. Blinova (Ref. 6), et al have studied the problem of formulating the solutions for and the reduction of equations such as (1.1), and also those having a more general form. It consists of reducing system (1.1) to a system of differential equations with constant coefficients. The latter is a problem which has not been entirely solved up to the present time.

This article will investigate the problem of determining a solution of the system (1.1) by compiling the reduction matrix. The

* Numbers given in the margin indicate pagination in the original foreign text.

Newtonian type of accelerated convergence method is employed to compile the reduction matrix; this method was developed and applied with great success in the studies of N. A. Kolmogorov (Ref. 10), V. I. Arnol'd (Ref. 2), and N. N. Bogolyubov (Ref. 4).

Just as in the study (Ref. 11), it is advantageous to introduce the correction $\xi = \xi_{ij}$ (i, j = 1, 2, ..., n) in the system of equations (1.1), and to investigate the following system of differential equations instead of it:

$$\frac{dx}{dt} = Ax + [P(\omega t, \xi) + \xi]\, x, \tag{1.2}$$

where $A + \xi = B$, A is the matrix of a linear system of differential equations with constant coefficients, which we may obtain after transformation of the system of equations (1.1), $P(\phi, \xi)$ -- the matrix which is periodic with respect to $\phi = (\phi_1, \ldots, \phi_m)$ with the period 2π and which is analytic with respect to the complex arguments ϕ, ξ in the region

$$|\mathrm{Im}\,\varphi| \leqslant \varrho, \qquad |\xi| \leqslant \sigma, \qquad |\xi| = \sum_{i,j=1}^{n} |\xi_{ij}|. \tag{1.3}$$

Let us pose the following problem: Let us find the transformation which is analytical with respect to ϕ /43

$$x = \Phi(\varphi)\, y \qquad (\varphi = \omega t), \tag{1.4}$$

where $\Phi(\phi)$ is the matrix which is periodic with respect to ϕ having the period 2π, and let us find $\xi = \xi^{(\infty)}$, so that system (1.2) may be reduced to a linear system with constant coefficients in the case of $\xi = \xi^{(\infty)}$

$$\frac{dy}{dt} = Ay. \tag{1.5}$$

Then, integrating system (1.5), we may obtain the general solution of system (1.2) in the form

$$x = \Phi(\omega t)\, e^{At} x_0. \tag{1.6}$$

The solution of this problem exists due to the results obtained in (Ref. 11). However, in our opinion it is of interest to perform a more detailed computation in the case under consideration of equation (1.2), due to the fact that the reduction of equations with quasi-periodic coefficients is a pressing problem. We shall employ the solution of the problem to solve not only the question of the reduction of system (1.2), but also to clarify the form of the solution for this system and to provide a method for compiling the reduction matrix $\Phi(\omega t)$

2. Before proceeding with the proof of the basic theorems for system (1.2), let us discuss the derivation of certain quantities which are necessary later on.

Let us employ $\lambda_1, \ldots, \lambda_n$ to designate the eigen values of the matrix A, and let us assume that the following inequality is fulfilled for certain constants $\varepsilon > 0$, $d > 0$

$$|\lambda_j - \lambda_l + i(k\omega)| \geqslant \varepsilon |k|^{-d} \qquad (j, l = 1, \ldots, n) \tag{2.1}$$

for any integral vectors $k = (k_1, \ldots, k_m)$, where $(k\omega) = k_1\omega_1 + \ldots + k_m\omega_m$, $|k| = \sum_{j=1}^{m} |k_j|$, $i = \sqrt{-1}$.

We shall show that when condition (2.1) is fulfilled, the equation

$$i(k\omega) Y_k = AY_k - Y_k A + P_k, \tag{2.2}$$

where

$$P_k = \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} P(\varphi) e^{-i(k\varphi)} d\varphi, \qquad |k| \geqslant 0, \tag{2.3}$$

may be solved for Y_k in the case of $|k| > 0$, and the following relationship is valid for Y_k

$$|Y_k| \leqslant c_0 |k|^{d_1} |P_k|, \tag{2.4}$$

in which c_0, d_1 $(d_1 \geqslant d)$ are certain positive constants which do not depend on k.

In order to obtain the relation (2.4), let us write the matrix A in the form

$$A = UJU^{-1}, \tag{2.5}$$

where J is the matrix in the Jordan form which may be written as follows

$$J = \{J_{\varrho_1}(\lambda_1), \ldots, J_{\varrho_l}(\lambda_l)\}, \tag{2.6}$$

where

$$J_{\varrho_j}(\lambda_j) = \lambda_j E_{\varrho_j} + Z_{\varrho_j}, \tag{2.7}$$

$E_{\varrho j}$---ϱ_j -dimensional unit matrix, $Z_{\varrho j}$---ϱ_j -dimensional matrix, /44 the subdiagonal of which consists of ones, and all the remaining elements consist of zeros:

$$E_{\varrho_j} = \delta_{\alpha\beta}^{(j)} = \begin{cases} 0, & \alpha \neq \beta, \\ 1, & \alpha = \beta \quad (\alpha, \beta = 1, \ldots, \varrho_j); \end{cases} \tag{2.8}$$

$$Z_{\varrho_j} = \delta_{\alpha-1,\beta}^{(j)} = \begin{cases} 0, & \alpha - 1 \neq \beta \quad (\alpha, \beta = 1, \ldots, \varrho_j), \\ 1, & \beta = \alpha - 1 \quad (\alpha = 2, \ldots, \varrho_j). \end{cases} \tag{2.9}$$

Substituting according to (2.5) the value of A in equation (2.2), we obtain

$$i(k\omega) Y_k = UJU^{-1}Y_k - Y_k UJU^{-1} + P_k$$

or

$$i(k\omega) U^{-1}Y_k U + U^{-1}Y_k U \cdot J = J \cdot U^{-1}Y_k U + U^{-1}P_k U. \tag{2.10}$$

Let us set

$$U^{-1}Y_k U = X, \qquad U^{-1}P_k U = Q; \tag{2.11}$$

and then equation (2.10) may be written in the following form

$$i(k\omega) X + XJ = JX + Q. \tag{2.12}$$

Let us now partition the matrices X and Q into blocks:

$$X = \{X_{\varrho_j\varrho_r}\}, \qquad Q = \{Q_{\varrho_j\varrho_r}\}, \tag{2.13}$$

where $X_{\varrho_j\varrho_r}$, $Q_{\varrho_j\varrho_r}$ $(j, r = 1, \ldots, l)$ are the matrices having ϱ_j rows and ϱ_r columns.

Equation (2.12) may be broken down into a system of independent equations

$$i(k\omega) X_{\varrho_j\varrho_r} + X_{\varrho_j\varrho_r} J_{\varrho_r}(\lambda_r) = J_{\varrho_j}(\lambda_j) X_{\varrho_j\varrho_r} + Q_{\varrho_j\varrho_r} \qquad (j, r = 1, \ldots, l). \tag{2.14}$$

Let us employ $x_\alpha^{(jr)}$, $q_\alpha^{(jr)}$ $(\alpha = 1, \ldots, \varrho_j)$ to designate the vectors

$$x_\alpha^{(jr)} = (x_{\alpha 1}^{(jr)}, \ldots, x_{\alpha\varrho_r}^{(jr)}), \qquad q_\alpha^{(jr)} = (q_{\alpha_1}^{(jr)}, \ldots, q_{\alpha\varrho_r}^{(jr)}), \tag{2.15}$$

which are the α-rows of the matrices $X_{\varrho_j\varrho_r}$, $Q_{\varrho_j\varrho_r}$.

Taking the notation in (2.7) into account, we may write the system (2.14) in the following form

$$x_\alpha^{(jr)} J_{\varrho_r}(\lambda_r - \lambda_j + i(k\omega)) = x_{\alpha-1}^{(jr)} + q_\alpha^{(jr)}, \tag{2.16}$$

where $x_0^{(jr)} = 0,\ \alpha = 1, \ldots, \varrho_j,\ j, r = 1, \ldots, l.$

According to the condition (2.1), it is apparent that system (2.16) may be solved. The solution for it has the following form

$$x_\alpha^{(jr)} = \{x_{\alpha-1}^{(jr)} + q_\alpha^{(jr)}\} J_{\varrho_r}^{-(jr)} (\lambda_r - \lambda_j + i(k\omega)), \tag{2.17}$$

where

$$J_{\varrho_r}^{-(jr)} (\lambda_r - \lambda_j + i(k\omega)) = \frac{1}{\lambda_r - \lambda_j + i(k\omega)} \left\{E_{\varrho_r} - \frac{Z_{\varrho_r}}{\lambda_r - \lambda_j + i(k\omega)} + \right.$$
$$\left. + \ldots + (-1)^{\varrho_r - 1} \frac{Z_{\varrho_r}^{\varrho_r - 1}}{(\lambda_r - \lambda_j + i(k\omega))^{\varrho_r - 1}}\right\}. \tag{2.18}$$

Let us now find the relationship for expressions (2.17).

According to (2.18) and (2.9), we have

$$|x_1^{(jr)}| \leqslant |q_1^{(jr)}| \left\{\frac{\varrho_r}{|\lambda_r - \lambda_j + i(k\omega)|} + \frac{\varrho_r - 1}{|\lambda_r - \lambda_j + i(k\omega)|^2} + \right.$$
$$\left. + \ldots + \frac{1}{|\lambda_r - \lambda_j + i(k\omega)|^{\varrho_r}}\right\}$$

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or, taking into account condition (2.1), we have

$$|x_1^{(jr)}| \leqslant |q_1^{(jr)}| (\varepsilon^{-1} |k|^d \varrho_r + \varepsilon^{-2} |k|^{2d} (\varrho_r - 1) + \ldots + \varepsilon^{-\varrho_r} |k|^{\varrho_r d}),$$
$$|x_1^{(jr)}| \leqslant c_1 \frac{|k|^{\varrho_r d}}{\varepsilon^{\varrho_r}} |q_1^{(jr)}|, \tag{2.19}$$

where c_1 is a constant which does not depend on k or ε.

Taking the inequality (2.19) into account, we may find the following for $\alpha = 2, 3, \ldots, \varrho_j$ from (2.17)

$$|x_\alpha^{(jr)}| \leqslant [|q_1^{(jr)}| + \ldots + |q_\alpha^{(jr)}|] c_1^\alpha \frac{|k|^{\alpha\varrho_r d}}{\varepsilon^{\alpha\varrho_r}}. \tag{2.20}$$

After this, taking into account (2.13), we obtain the following relationship for X:

$$|X| \leqslant c_2 \frac{|k|^{d_1}}{\varepsilon^{d_1/d}} |Q|, \tag{2.21}$$

where

$$d_1 = \max_{1<j<l} \varrho_j \max_{1<r<l} \varrho_r \cdot d,$$

c_2 is a positive constant which depends neither on k or on ε.

Turning from F and Q to Y_k and P_k, respectively, according to formulas (2.11) we obtain the inequality

$$|Y_k| \leqslant |U|^2 |U^{-1}|^2 c_2 \frac{|k|^{d_1}}{\varepsilon^{d_1/d}} |P_k|. \tag{2.22}$$

Setting in (2.22)

$$c_0 = |U|^2 |U^{-1}|^2 \frac{c_2}{\varepsilon^{d_1/d_2}}, \tag{2.23}$$

we obtain the desired inequality (2.4).

Let us now compile the following series from the solution of equations (2.2)

$$Y(\varphi) = \sum_{|k| \neq 0} Y_k e^{i(k\varphi)}. \tag{2.24}$$

Differentiating it, we find that Y(ϕ) is the formal solution of equation

$$\frac{\partial Y}{\partial \varphi} \omega + YA = AY + P(\varphi) - \overline{P(\varphi)}, \tag{2.25}$$

where

$$\overline{P(\varphi)} = P_0 = \frac{1}{(2\pi)^m} \int_0^{2\pi} \dots \int_0^{2\pi} P(\varphi)\, d\varphi. \tag{2.26}$$

3. The iteration process with accelerated convergence will be employed to compile the reduction matrix Φ(ϕ), which -- in view of the transformation (1.4) -- changes the system of equations (1.2) into a system (1.5) which may be integrated. Let us write the s^th^ (s ⩾ 1) step of this process. For this purpose, let us employ M_{s-1}, δ_{s-1}, ϱ_{s-1} to designate the constants by means of which we shall characterize the s - 1^th^ iteration, and let us employ M_s, δ_s, ϱ_s to designate the constants by means of which we shall characterize the s^th^ iteration.

The change from the s - 1th iteration to the sth iteration may be /46
characterized as follows.

Theorem 1. *Let us set for the system of equations*

$$\frac{dx}{dt} = Ax + [P(\varphi, \xi) + \xi]x,$$
$$\frac{d\varphi}{dt} = \omega \qquad (\omega = (\omega_1, \ldots, \omega_m)) \tag{3.1}$$

A, ξ *are the* n-*dimensional constant matrices,* P(ϕ, ξ) -- *the* n-*dimensional matrix which is periodic with respect to* $\phi = (\phi_1, \ldots, \phi_m)$ *having the period* 2π, *and analytical with respect to the complex arguments* ϕ, ξ *in the region*

$$|\mathrm{Im}\,\varphi| < \varrho_{s-1}, \qquad |\xi| < M_{s-2} \tag{3.2}$$

and which satisfies the inequality

$$|P(\varphi, \xi)| \leqslant M_{s-1}. \tag{3.3}$$

In addition, the eigen values of the matrix A *satisfy the inequality* (2.1).

For a sufficiently small m_0 *and any* $s \geqslant 1$, *we have the transformation*

$$x = [E + Y(\varphi, \xi_1)]x_1,$$
$$\xi = \xi(\xi_1), \tag{3.4}$$

which is periodic with respect to ϕ *having the period* 2π, *and analytical with respect to* ϕ, ξ *in the region*

$$|\mathrm{Im}\,\varphi| \leqslant \varrho_s, \qquad |\xi_1| \leqslant M_{s-1}, \tag{3.5}$$

which reduces the system of equations (3.1) *to the following form*

$$\frac{dx_1}{dt} = Ax_1 + [P_1(\varphi, \xi_1) + \xi_1]x_1,$$
$$\frac{d\varphi}{dt} = \omega, \tag{3.6}$$

where ξ_1 *is a constant matrix, and* $P_1(\phi, \xi_1)$ *is a matrix which is periodic with respect to* ϕ *having the period* 2π; *these matrices are analytical with respect to* ϕ, ξ *in the region* (3.5), *so that in the* *region* (3.5) *the following inequalities are valid*

$$|P_1(\varphi, \xi_1)| \leqslant M_s, \tag{3.7}$$

$$|Y(\varphi, \xi_1)| \leqslant \frac{M_{s-1}^{\varkappa-1}}{4n}, \qquad |\xi(\xi_1) - \xi_1| \leqslant M_{s-1}. \tag{3.8}$$

Thus, the constants M_s, δ_s, ϱ_s *are related to the constants* M_{s-1}, δ_{s-1}, ϱ_{s-1} *by the following relationships*

$$M_s = M_{s-1}^{\varkappa}, \qquad \delta_s = \gamma\delta_{s-1}, \qquad \varrho_s = \varrho_{s-1} - 2\delta_{s-1} \qquad (s \geqslant 1), \tag{3.9}$$

where

$$1 < \varkappa < 2, \qquad \gamma = \frac{\varrho_0}{4 + \varrho_0}, \qquad \delta_0 = \gamma, \qquad \varrho_0 = \varrho, \qquad M_{-1} = \sigma. \tag{3.10}$$

Proof. Let us first reduce the system (3.1) to the form (3.6). For this purpose, let us select the transformation of the coordinates in the form

$$x = [E + Y(\varphi, \xi)]\, x_1, \tag{3.11}$$ /47

where $Y(\phi, \xi)$ is the solution of equation (2.25).

Substituting (3.11) in equation (3.1) and taking (2.25) into account, we obtain the following system of equations:

$$\frac{dx_1}{dt} = Ax_1 + [P_1^{(1)}(\varphi, \xi) + \xi + \overline{P(\varphi, \xi)}]\, x_1, \qquad \frac{d\varphi}{dt} = \omega, \tag{3.12}$$

where we set

$$P_1^{(1)}(\varphi, \xi) = (E + Y)^{-1}[(P + \xi) Y - Y(\overline{P} + \xi)]. \tag{3.13}$$

Let us define the transformation $\xi = \xi(\xi_1)$ as the solution of the equation

$$\xi + \overline{P(\varphi, \xi)} = \xi_1. \tag{3.14}$$

Then the substitution of the variables (3.11) and (3.14) reduces the initial equation (3.1) to our desired form

$$\frac{dx_1}{dt} = Ax_1 + [P_1(\varphi, \xi_1) + \xi_1] x_1,$$

$$\frac{d\varphi}{dt} = \omega,$$

where

$$P_1(\varphi, \xi_1) = P_1^{(1)}(\varphi, \xi(\xi_1)). \tag{3.15}$$

Let us first examine equation (3.14). According to the condition of the theorem, the matrix $\overline{P(\phi, \xi)}$ is the analytical function of $\xi = (\xi_{\alpha\beta})$, $(\alpha, \beta = 1, \ldots, n)$ and satisfies the following inequality in the region (3.2)

$$\overline{|P(\varphi, \xi)|} \leqslant M_{s-1}. \tag{3.16}$$

It follows from the properties of the analytical functions of (3.2) and (3.16) that in the region

$$|\xi| \leqslant \frac{M_{s-2}}{2} \tag{3.17}$$

the following inequality is valid

$$\sum_{\alpha,\beta=1}^{n} \left| \frac{\overline{\partial P(\varphi, \xi)}}{\partial \xi_{\alpha\beta}} \right| \leqslant \frac{2M_{s-1}n^2}{M_{s-2}} \leqslant 2n^2 M_{s-2}^{\varkappa-1} \leqslant \frac{1}{2}. \tag{3.18}$$

from which it follows that equations (3.15) may be solved and $\xi = \xi(\xi_1)$ is the analytical function of ξ_1 in the region

$$|\xi_1| \leqslant M_{s-1},$$

and in this region we have the following inequality according to (3.15) and (3.16)

$$|\xi(\xi_1)| \leqslant 2M_{s-1} \leqslant \frac{M_{s-2}}{2},$$
$$|\xi(\xi_1) - \xi_1| \leqslant M_{s-1}. \tag{3.18'}$$

Let us derive one relation for ξ. Differentiating the equation

$$\xi(\xi_1) + \overline{P(\varphi, \xi(\xi_1))} = \xi_1,$$

we have /48

$$\frac{\partial \xi_{kq}}{\partial \xi_{ij}^{(1)}} + \sum_{\alpha,\beta} \frac{\overline{\partial P(\varphi, \xi)}}{\partial \xi_{\alpha\beta}} \cdot \frac{\partial \xi_{\alpha\beta}}{\partial \xi_{ij}^{(1)}} = \frac{\partial \xi_{kq}^{(1)}}{\partial \xi_{ij}^{(1)}} = \begin{cases} 1, & k = i, \quad q = j, \\ 0, & k \neq i \text{ or } q \neq j, \end{cases}$$

Taking (3.18) into account, we find

$$\left| \frac{\partial \xi_{kq}}{\partial \xi_{ij}^{(1)}} \right| \leqslant \left| \frac{\partial \xi_{kq}^{(1)}}{\partial \xi_{ij}^{(1)}} \right| + 2n^2 M_{s-2}^{\varkappa-1} \max_{kq} \left| \frac{\partial \xi_{kq}}{\partial \xi_{ij}^{(1)}} \right|$$

or, performing summation over i, j, we obtain

$$\max_{kq} \sum_{ij} \left| \frac{\partial \xi_{kq}}{\partial \xi_{ij}^{(1)}} \right| \leqslant \frac{1}{1 - 2n^2 M_{s-2}^{\varkappa-1}} \leqslant 1 + 4n^2 M_{s-2}^{\varkappa-1} \leqslant 2. \tag{3.19}$$

Let us now determine the matrix Y (ϕ,ξ) included in the formula for the substitution of the variables (3.11).

According to (2.4) and the notation of (2.24), we have the following inequality

$$|Y(\varphi, \xi)| \leqslant \sum_{|k| \neq 0} |Y_k| |e^{i(k\varphi)}| \leqslant c_0 \sum_{|k| \neq 0} |k|^{d_1} |P_k(\xi)| e^{|\operatorname{Im} \varphi||k|} \tag{3.20}$$

for all ϕ, ξ from the region (3.5). However, the function P(ϕ,ξ) is analytical and confined in the region (3.2). Therefore, the following relationship is valid for its Fourier coefficients $P_k(\xi)$

$$|P_k(\xi)| < M_{s-1} e^{-\varrho_{s-1}|k|}.$$

If we take this into account, we obtain the following inequality for $|Y(\phi,\xi)|$

$$|Y(\varphi, \xi)| \leqslant c_0 M_{s-1} \sum_{|k| \neq 0} |k|^{d_1} e^{(|\operatorname{Im} \varphi| - \varrho_{s-1})|k|}. \tag{3.21}$$

It follows from (3.21) for $|\operatorname{Im}\phi| \leqslant \varrho_s$

$$\begin{aligned} |Y(\varphi, \xi)| &\leqslant c_0 M_{s-1} \sum_{|k| \neq 0} |k|^{d_1} e^{-2\delta_{s-1}|k|} \leqslant c_0 \left(\frac{d_1}{e}\right)^{d_1} \frac{(1+e)^m}{\delta_{s-1}^{d_1+m}} M_{s-1} = \\ &= c_0 \left(\frac{d_1}{e}\right)^{d_1} (1+e)^m \frac{M_{s-1}^{2-\varkappa}}{\gamma^{s(d_1+m)}} M_{s-1}^{\varkappa-1} \leqslant \frac{M_{s-1}^{\varkappa-1}}{8n} < \frac{M_{s-1}^{\varkappa-1}}{4n}. \end{aligned} \tag{3.22}$$

as soon as M_0 is sufficiently small. Thus, all the relations (3.8) have been substantiated.

The analytical nature of the function Y (ϕ, $\xi(\xi_1)$) with respect to ξ_1 in the region $|\xi_1| < M_{s-1}$ follows from the analytical nature of the function $P(\phi,\xi(\xi_1))$, taking (2.24) into account. Its analytical nature with respect to ϕ in the region $|\text{Im}\phi| \leqslant \varrho_s$ follows from

$$\left|\frac{\partial Y(\varphi,\xi)}{\partial\varphi}\right| \leqslant \sum_{|k|\neq 0} |Y_k|\,|k|\,e^{\varrho_s|k|} \leqslant c_0 M_{s-1} \sum_{|k|\neq 0} |k|^{d_1+1} e^{-2\delta_{s-1}|k|} \leqslant$$
$$\leqslant c_0\left(\frac{d_1+1}{e}\right)^{d_1+1}(1+e)^m \frac{M_{s-1}^{\varkappa}}{\gamma^{s(d_1+m+1)}} \leqslant c_1 M_{s-1}^{\varkappa-1}.$$

Let us now turn to the relationship for (3.15). We have

$$|P_1(\varphi,\xi_1)| = |P_1^{(1)}(\varphi,\xi(\xi_1))| \leqslant |(E+Y(\varphi,\xi))^{-1}| \times$$
$$\times\{|P(\varphi,\xi)+\xi(\xi_1)|\cdot|Y(\varphi,\xi)|+|Y(\varphi,\xi)|\,|\xi_1|\}. \tag{3.23}$$

Therefore, taking the inequalities (3.3), (3.18') and (3.22) into account, we obtain /49

$$|P_1(\varphi,\xi_1)| \leqslant \sum_{k=0}^{\infty}\left(\frac{M_{s-1}}{4n}\right)^k\cdot 4M_{s-1}\,\frac{M_{s-1}^{\varkappa-1}}{8n} \leqslant \frac{1}{n}M_{s-1}^{\varkappa} \leqslant M_s,$$

which concludes the proof for theorem 1.

4. Let us now apply the preceding theorem to construct the iteration process with accelerated convergence for system (1.2), which would successively increase the smallness of the function P (ϕ, ξ) and would simultaneously provide an explicit expression for the reduction matrix $\Phi(\phi)$.

We shall assume

$$x = [E+U^{(1)}(\varphi,\xi)]\,x^{(1)},$$
$$\xi = \xi(\xi^{(1)}), \tag{4.1}$$

where $U^{(1)}$ (ϕ,ξ) = Y (ϕ,ξ) is determined from equation (2.25) and satisfies the inequalities (3.8) in the case of s = 1.

In this case, $x^{(1)}$, ϕ will satisfy equations

$$\frac{dx^{(1)}}{dt} = Ax^{(1)} + [P^{(1)}(\varphi, \xi^{(1)}) + \xi^{(1)}]\, x^{(1)},$$
$$\frac{d\varphi}{dt} = \omega. \qquad (4.2)$$

Let us again apply the transformation (which corresponds to this new system) according to theorem 1 to equations (4.2). As a result, we obtain

$$x^{(1)} = [E + U^{(2)}(\varphi, \xi^{(1)})]\, x^{(2)},$$
$$\xi^{(1)} = \xi^{(1)}(\xi^{(2)}). \qquad (4.3)$$

where $U^{(2)}(\phi, \xi^{(1)})$ -- the solution for an equation such as (2.5), accordingly compiled for systems (4.2), and $x^{(2)}$, ϕ -- a solution of the system of equations:

$$\frac{dx^{(2)}}{dt} = Ax^{(2)} + [P^{(2)}(\varphi, \xi^{(2)}) + \xi^{(2)}]\, x^{(2)},$$
$$\frac{d\varphi}{dt} = \omega. \qquad (4.4)$$

Continuing this process, at the $s^{\underline{th}}$ step we obtain

$$x^{(s-1)} = [E + U^{(s)}(\varphi, \xi^{(s-1)})]\, x^{(s)},$$
$$\xi^{(s-1)} = \xi^{(s-1)}(\xi^{(s)}), \qquad (4.5)$$

where $x^{(s)}$, ϕ satisfy the equations

$$\frac{dx^{(s)}}{dt} = Ax^{(s)} + [P^{(s)}(\varphi, \xi^{(s)}) + \xi^{(s)}]\, x^{(s)},$$
$$\frac{d\varphi}{dt} = \omega. \qquad (4.6)$$

The continuation of the iteration process for all $s \geqslant 1$ is provided by the choice of parameters (3.9) and by the statements of theorem 1. Therefore, let us employ the relationship of theorem 1 to prove the convergence of this process. /50

Let us first express x, ξ by $x^{(s)}$ and $\xi^{(s)}$. According to the formulas for the substitution of variables (4.1), (4.3), (4.5), we have

$$x = [E + U^{(1)}(\varphi, \xi(\xi^{(1)}))] \dots [E + U^{(s)}(\varphi, \xi^{(s-1)}(\xi^{(s)}))]\, x^{(s)},$$
$$\xi = \xi(\xi^{(1)} \dots (\xi^{(s)})), \qquad (4.7)$$

Taking the fact into account that $\xi^{(j)}(\xi^{(j+1)}) = \xi^{(j)}(\xi^{(j+1)} \ldots (\xi^{(s)}))$, we may find the expression for x and ξ in terms of $x^{(s)}$ and $\xi^{(s)}$:

$$x = \prod_{j=1}^{s} [E + U^{(j)}(\varphi, \xi^{(j-1)}(\ldots(\xi^{(s)})))]\, x^{(s)},$$
$$\xi = \xi(\xi^{(1)} \ldots (\xi^{(s)})), \tag{4.8}$$

while in (4.8) the product of the matrices must be taken in the order determined by formula (4.7).

Let us first investigate the sequence

$$B^{(s)}(\xi^{(s)}) = \xi(\xi^{(1)} \ldots (\xi^{(s)})) \qquad (s = 1, 2, \ldots). \tag{4.9}$$

Since $\xi^{(j)}(\xi^{(j+1)})$ for $0 \leqslant j \leqslant s - 1$ is the solution of equation

$$\xi^{(j)}(\xi^{(j+1)}) + \overline{P^{(j)}(\varphi, \xi^{(j)}(\xi^{(j+1)}))} = \xi^{(j+1)}, \tag{4.10}$$

in which $|P^{(j)}(\phi,\xi^{(j)})| \leqslant M_j$, taking (3.19) into account, we may write

$$\sum_{\alpha,\beta} \left| \frac{\partial \xi^{(j)}}{\partial \xi^{(j+1)}_{\alpha\beta}} \right| \leqslant 1 + 4n^2 M_{j-2}^{\varkappa-1} \tag{4.11}$$

for $|\xi^{(j+1)}| \leqslant M_j$. However, in view of the relationship (3.18') it follows from inequality $|\xi^{(s)}| \leqslant M_{s-1}$ that

$$|\xi^{(s-1)}(\xi^{(s)})| \leqslant 2M_{s-1} \leqslant \frac{M_{s-2}}{2},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$|\xi^{(j)}(\xi^{(j+1)} \ldots (\xi^{(s)}))| \leqslant 2M_j \leqslant \frac{M_{j-1}}{2},$$

from which we may conclude that in the case of $|\xi^{(s)}| \leqslant M_{s-1}$ inequality (4.11) remains in force for $\xi^{(j)} = \xi^{(j)}(\xi^{(j+1)} \ldots (\xi^{(s)}))$.

Therefore, differentiating (4.9), we obtain the relationship

$$\sum_{\alpha,\beta} \left| \frac{\partial B^{(s)}(\xi^{(s)})}{\partial \xi^{(s)}_{\alpha\beta}} \right| \leqslant (1 + 4n^2\sigma^{\varkappa-1})(1 + 4n^2 M_0^{\varkappa-1}) \ldots (1 + 4n^2 M_{s-2}^{\varkappa-1}) \leqslant$$
$$\leqslant 2 \prod_{j=0}^{\infty} (1 + 4n^2 M_j^{\varkappa-1}) \leqslant c_1, \tag{4.12}$$

after which we may write

$$|B^{(s+1)}(0)-B^{(s)}(0)|=|B^{(s)}(\xi^{(s)}(0))-B^{(s)}(0)|\leqslant c_1|\xi^{(s)}(0)|\leqslant c_1M_{s-1}. \quad (4.13)$$

Employing (4.13), we may find the relationship

$$|B^{(s+k)}(0)-B^{(s)}(0)|\leqslant c_1\sum_{i=0}^{k-1}M_{s+i-1}\leqslant c_2M_{s-1}, \quad (4.14)$$

according to which the uniform convergence of the matrix $B^{(s)}(0)$ is apparent: /51

$$B^{(s)}(0)\to\xi^{(\infty)} \text{ for } s\to\infty, \quad (4.15)$$

while

$$|\xi^{(\infty)}-B^{(s)}(0)|\leqslant c_2M_{s-1}\leqslant M_{s-2}^{\varkappa-1}. \quad (4.16)$$

Let us now turn to the proof of the uniform convergence of the sequence

$$\Phi^{(s)}(\varphi,\xi^{(s)})=\prod_{i=1}^{s}[E+U^{(i)}(\varphi,\xi^{(i-1)}\dots(\xi^{(s)}))]. \quad (4.17)$$

First of all, we shall show that the sequences $\xi^{(j)}(\dots(\xi^{(s-1)}(0)))$ ($j = 0, 1, \dots, s-1$) converge uniformly in the case of $s\to\infty$. By definition, $\xi^{(j)}(\xi^{(j+1)})$ is the solution of the equation

$$\xi^{(j+1)}=\xi^{(j)}+\overline{P^{(j)}(\varphi,\xi^{(j)})} \qquad (\xi^{(0)}=\xi^{(\infty)}).$$

Therefore

$$\xi^{(j+1)}=\xi^{(j)}(\xi^{(j+1)})+\overline{P^{(j)}(\varphi,\xi^{(j)}(\xi^{(j+1)}))}$$

holds identically, from which we obtain

$$\xi^{(j+1)}(\dots\xi^{(s-1)}(0))=\xi^{(j)}(\xi^{(j+1)}\dots(\xi^{(s-1)}(0)))+\overline{P^{(j)}(\varphi,\xi^{(j)}(\dots\xi^{(s-1)}(0)))}. \quad (4.18)$$

Since $\xi(\xi^{(1)}\dots(\xi^{(s-1)}(0)))\underset{s\to\infty}{\to}\xi^{(\infty)}$ in view of (4.15), going to the limit in (4.18), we obtain

$$\xi_0^{(1)} = \lim_{s\to\infty} \xi^{(1)} (\xi^{(2)} \ldots (\xi^{(s-1)} (0))) = \xi^{(\infty)} + \overline{P^{(1)} (\varphi, \xi^{(\infty)})},$$

. . . .

$$\xi_0^{(j+1)} = \lim_{s\to\infty} \xi^{(j+1)} (\xi^{(j+2)} \ldots (\xi^{(s-1)} (0))) = \xi_0^{(j)} + \overline{P^{(j)} (\varphi, \xi_0^{(j)})}, \tag{4.19}$$

and the following inequalities are retained for $\xi_0^{(j)}$

$$|\xi_0^{(j)}| \leqslant 2M_j \leqslant \frac{M_{j-1}}{2} \tag{4.20}$$

and the following equation holds

$$\xi^{(j)} (\xi^{(j+1)} (0)) = \xi_0^{(j)}. \tag{4.21}$$

Assuming that $\xi^{(s)} = \xi_0^{(s)}$ in (4.17) and taking (4.21) into account, we find that

$$\Phi^{(s)} (\varphi, \xi_0^{(s)}) = \prod_{j=1}^{s} [E + U^{(j)} (\varphi, \xi_0^{(j-1)})]. \tag{4.22}$$

Taking the fact into account that

$$|U^{(j)} (\varphi, \xi_0^{j-1})| \leqslant \frac{M_{j-1}^{\varkappa-1}}{4n}, \tag{4.23}$$

we obtain /52

$$|\Phi^{(s+1)} (\varphi, \xi_0^{(s+1)}) - \Phi^{(s)} (\varphi, \xi_0^{(s)})| \leqslant |\Phi^{(s)} (\varphi, \xi_0^{(s)})| \, |U^{(s)} (\varphi, \xi_0^{(s-1)})| \leqslant$$
$$\leqslant \left| \prod_{j=1}^{s} \left(E + \frac{M_{j-1}^{\varkappa-1}}{4n} I \right) \right| \frac{M_{s-1}^{\varkappa-1}}{4n} \leqslant \frac{n}{4} \prod_{j=1}^{\infty} \left(1 + \frac{M_j^{\varkappa-1}}{4} \right) M_{s-1}^{\varkappa-1} \leqslant c_3 M_{s-1}^{\varkappa-1}, \tag{4.24}$$

where I is the matrix, all of whose members equal unity.

Employing (4.24), we find the relationship

$$|\Phi^{(s+k)} (\varphi, \xi_0^{(s+k)}) - \Phi^{(s)} (\varphi, \xi_0^{(s)})| \leqslant c_3 \sum_{j=0}^{k-1} M_{s+j-1}^{\varkappa-1} \leqslant c_4 M_{s-1}^{\varkappa-1}, \tag{4.25}$$

according to which the uniform convergence of $\Phi^{(s)}(\phi, \xi_0^{(s)})$ is apparent:

$$\Phi^{(s)} (\varphi, \xi_0^{(s)}) \to \Phi (\varphi) \text{ for } s \to \infty, \tag{4.26}$$

while

$$|\Phi^{(s)}(\varphi, \xi_0^{(s)}) - \Phi(\varphi)| \leqslant c_4 M_{s-1}^{\varkappa-1}. \tag{4.27}$$

The functions $\Phi^{(s)}(\phi, \xi_0^{(s)})$ are analytical functions of ϕ for $|\mathrm{Im}\phi| \leqslant \rho_s$, and consequently $\Phi(\phi)$ is the analytical function of ϕ for $|\mathrm{Im}\,\varphi| \leqslant \varrho_0 - 2\sum_{s=0}^{\infty}\delta_s = \frac{\varrho_0}{2}$. We shall show that the matrix $\Phi(\phi)$ is not degenerate. According to (4.22) and (4.23), the following inequality holds

$$|\Phi^{(s)}(\varphi, \xi_0^{(s)}) - E| = \left|\prod_{j=1}(E + U^{(j)}(\varphi, \xi_0^{(j-1)})) - E\right| \leqslant$$

$$\leqslant \left|\left(\frac{M_0^{\varkappa-1}}{4n} + \dots + \frac{M_{s-1}^{\varkappa-1}}{4n}\right) I + \left(\frac{M_0^{\varkappa-1}}{4n}\cdot\frac{M_1^{\varkappa-1}}{4n} n + \dots + \frac{M_{s-2}^{\varkappa-1}}{4n}\cdot\frac{M_{s-1}^{\varkappa-1}}{4n} n\right) I +\right.$$

$$\left. + \dots + \frac{M_0^{\varkappa-1}}{4n}\cdot\frac{M_1^{\varkappa-1}}{4n}\dots\frac{M_{s-1}^{\varkappa-1}}{4n}\cdot n^{s-1} I\right| \leqslant \left[\prod_{j=1}^{s}\left(1 + \frac{M_{j-1}^{\varkappa-1}}{4}\right) - 1\right] |I| \leqslant c_5 < 1,$$

from which it follows that

$$|\Phi(\varphi) - E| \leqslant c_5 < 1. \tag{4.28}$$

When (4.28) is fulfilled, the series $\sum_{j=0}^{\infty}(E - \Phi(\varphi))^j$ converges and determines the matrix which is the inverse of $\Phi(\phi)$:

$$\sum_{j=0}^{\infty}(E - \Phi)^j = [E - (E - \Phi)]^{-1} = \Phi^{-1}; \tag{4.29}$$

This indicates that $\Phi(\phi)$ is a nondegenerate matrix.

Summing up the results obtained above, let us turn to the following theorem.

Theorem 2. *Let the matrix* $P(\phi, \xi)$ *be periodic with respect to* $\phi = (\phi_1, \dots, \phi_m)$ *with the period* 2π *and analytic with respect to* ϕ, $\xi = \{\xi_{\alpha\beta}\}$ $(\alpha, \beta = 1, \dots, n)$ *in the region*

$$|\mathrm{Im}\,\varphi| \leqslant \varrho_0, \qquad |\xi| = \sum_{\alpha,\beta}|\xi_{\alpha\beta}| \leqslant \sigma. \tag{4.30}$$

In addition, $\lambda_1, \dots, \lambda_n$ *are the eigen values of the matrix* A. /53

Let us assume that the following inequality is fulfilled for certain $\varepsilon > 0$, $d > 0$

$$|\lambda_j - \lambda_l + i(k\omega)| \geqslant \varepsilon |k|^{-d} \tag{4.31}$$

for all integral vectors $k = (k_1, \ldots, k_m)$, *where*

$$j, l = 1, \ldots, n, \qquad (k\omega) = k_1\omega_1 + \ldots + k_m\omega_m, \qquad |k| = \sum_{i=1}^{m} |k_i|.$$

It is then possible to take a small enough positive constant $M_0 = M_0(\varepsilon, d)$ *and the matrix* ξ, $|\xi| < 2M_0^{\varkappa-1}$ $(1 < \varkappa < 2)$ *so that for*

$$|P(\varphi, \xi)| \leqslant M_0 \tag{4.32}$$

the system of equations

$$\frac{dx}{dt} = Ax + [P(\varphi, \xi) + \xi]x, \qquad \frac{d\varphi}{dt} = \omega \tag{4.33}$$

with substitution of the variables

$$x = \Phi(\varphi)y \tag{4.34}$$

with the matrix $\Phi(\phi)$ *which is periodic with respect to* ϕ *of the period* 2π, *which is also analytic and has an analytical inverse matrix in the region*

$$|\operatorname{Im}\varphi| \leqslant \frac{\varrho}{2} \tag{4.35}$$

may be reduced to the following form

$$\frac{dy}{dt} = Ay, \qquad \frac{d\varphi}{dt} = \omega. \tag{4.36}$$

Thus, the matrix $\Phi(\phi)$ *may be represented in the form of the product*

$$\Phi(\varphi) = \prod_{j=1}^{\infty} (E + U^{(j)}(\varphi)), \tag{4.37}$$

where

$$U^{(j)}(\varphi) = \sum_{|k| \neq 0} Y_k^{(j)} e^{i(k\varphi)}. \tag{4.38}$$

$Y_k^{(j)}$ *is the solution of the equation*

$$i(k\omega) Y_k^{(j)} = AY_k^{(j)} - Y_k^{(j)}A + P_k^{(j-1)}. \tag{4.39}$$

in which $P_k^{(j)} = (2\pi)^{-m} \int\limits_0^{2\pi} \dots \int\limits_0^{2\pi} P^{(j)}(\varphi) e^{-i(k\varphi)} d\varphi_1 \dots d\varphi_m$ *are the Fourier coefficients of the function* $P^{(j)}(\phi)$, *which is determined by the relationships*

$$P^{(j)} = [E + U^{(j)}]^{-1} [P^{(j-1)} U^{(j)} - U^{(j)} \overline{P^{(j-1)}}],$$

$$P^{(0)} = P(\varphi, \xi) + \xi, \qquad \overline{P^{(j-1)}} = (2\pi)^{-m} \int\limits_0^{2\pi} \dots \int\limits_0^{2\pi} P^{(j-1)}(\varphi) \, d\varphi_1 \dots d\varphi_m, \tag{4.40}$$

and satisfies the inequality /54

$$\left| \Phi(\varphi) - \prod_{j=1}^{s} [E + U^{(j)}(\varphi)] \right| \leqslant 2M_0^{(\varkappa-1)\varkappa^{s-1}}. \tag{4.41}$$

5. Let us now turn to the inequality

$$|\lambda_j - \lambda_l + i(k\omega)| \geqslant \varepsilon |k|^{-d}. \tag{5.1}$$

The following relationship is a necessary condition for its fulfillment:

$$|(k\omega)| \geqslant \varepsilon |k|^{-d}. \tag{5.2}$$

Condition (5.2) is sufficient for matrices A with real eigen values. The result of theorem 2 may be intensified for systems with these matrices. Namely, the following theorem is applicable.

Theorem 3. *Let* $P(\phi)$ *be a matrix which is periodic with respect to* $\phi = (\phi_1, \dots, \phi_m)$ *with the period* 2π *and analytic in the region*

$$|\mathrm{Im}\,\varphi| \leqslant \varrho_0 \tag{5.3}$$

In addition, let us assume that the eigen values of the matrix A are real and different.

Let us assume that for certain $\varepsilon > 0$, $d > 0$ *the following inequality is fulfilled*

$$|(k\omega)| \geqslant \varepsilon |k|^{-d} \tag{5.4}$$

for all integral vectors $k = (k_1, \dots, k_m)$.

It is then possible to take a small enough positive constant M_0 *so that for*

$$|P(\varphi)| \leqslant M_0 \tag{5.5}$$

the system of equations

$$\frac{dx}{dt} = Ax + P(\varphi)x, \qquad \frac{d\varphi}{dt} = \omega \tag{5.6}$$

with substitution of the variables

$$x = \Phi(\varphi)y \tag{5.7}$$

with the matrix $\Phi(\phi)$ *which is periodic with respect to* ϕ *of the period* 2π*, and which is analytic and has an analytical inverse matrix in the region*

$$|\operatorname{Im}\varphi| \leqslant \frac{\varrho_0}{2} \tag{5.8}$$

may be reduced to the following form

$$\frac{dy}{dt} = A^0 y, \qquad \frac{d\varphi}{dt} = \omega. \tag{5.9}$$

In addition, the matrices $\Phi(\phi)$ *and* A^0 *may be represented in the following form*

$$\Phi(\varphi) = \prod_{j=1}^{\infty}(E + U^{(j)}(\varphi)), \qquad A^0 = \sum_{j=1}^{\infty}\overline{P^{(j-1)}(\varphi)} + A, \tag{5.10}$$

where

$$U^{(j)}(\varphi) = \sum_{|k|\neq 0} Y_k^{(j)} e^{i(k\varphi)}, \tag{5.11}$$

$Y_k^{(j)}$ *is the solution of the equation* /55

$$i(k\omega)Y_k^{(j)} = \left[A + \sum_{\alpha=0}^{j-1}\overline{P^{(\alpha)}(\varphi)}\right]Y_k^{(j)} - Y_k^{(j)}\left[A + \sum_{\alpha=0}^{j-1}\overline{P^{(\alpha)}(\varphi)}\right] + P_k^{(j-1)}, \tag{5.12}$$

in which $P_k^{(j)} = (2\pi)^{-m}\int_0^{2\pi}\dots\int_0^{2\pi} P^{(j)}(\varphi)e^{-i(k\varphi)}d\varphi_1\dots d\varphi_m$, $\overline{P^{(j)}(\varphi)} = P_0^{(j)}$ *are the Fourier coefficients of the function* $P^{(j)}(\phi)$*, which is determined by relationships*

$$P^{(j)} = (E + U^{(j)})^{-1} [P^{(j-1)} U^{(j)} - U^{(j)} \overline{P^{(j-1)}}], \quad (5.13)$$

$$P^{(0)} = P(\varphi);$$

*We thus have**

$$\left| \Phi(\varphi) - \prod_{j=1}^{s} (E + U^{(j)}(\varphi)) \right| \leqslant 2M_0^{(\varkappa-1)\varkappa^{s-1}},$$

$$\left| \sum_{j=s}^{\infty} \overline{P^{(j)}(\varphi)} \right| \leqslant 2M_0^{\varkappa^s}. \quad (5.14)$$

Proof. First of all, let us establish one set of matrices with real and different eigen values. We shall prove the following lemma in this connections.

Lemma. *Let us assume that the eigen values* $\lambda_1, \ldots, \lambda_n$ *of the matrix A are real and different. Let us employ* $\mathfrak{A}_r$ *to designate the set of matrices A which satisfy the inequality*

$$|A - \overline{A}| = \sum_{i,j=1}^{n} |a_{ij} - \overline{a}_{ij}| \leqslant r. \quad (5.15)$$

We may then take $r = r_0(\lambda_1, \ldots, \lambda_n) > 0$ *in such a way that all the matrices* $\overline{A}$ *of* $\mathfrak{A}_{r_0}$ *have real and different eigen values.*

Actually, let us enclose $\lambda_1, \ldots, \lambda_n$ in nonintersecting segments $I_1, \ldots, I_n$, which have their centers at the points $\lambda_1, \ldots, \lambda_n$, respectively.

The function $f(\lambda)=\det|A-\lambda E|$ continuously represents each of the segments I_j in the segment $f(I_j) = (\underline{m}_j, \overline{m}_j)$, where $\underline{m}_j = \min_{\lambda \in I_j} f(\lambda)$, $\overline{m}_j = \max_{\lambda \in I_j} f(\lambda)$. Thus, since $\lambda_j \neq \lambda_r$, for $j \neq r$, we then have

$$\underline{m}_j < 0, \qquad \overline{m}_j > 0. \quad (5.16)$$

Let us investigate the function

$$f_r(\lambda) = \det|A + rA_1 - \lambda E|, \quad (5.17)$$

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* The reduction of a system like (5.6) is proven on the basis of these same assumptions in studies by L. Ya. Adrianova (Ref. 3) and I. N. Blinov (Ref. 6). However, it is proven by another method.

where A_1 is any matrix for which $|A_1| \leqslant 1$.

The function $f_r(\underline{\lambda})$ continuously represents each of the segments I_j in the segment $f_r(I_j) = [\underline{m}_j(r), \overline{m}_j(r)]$, where

$$\underline{m}_j(r) = \min_{\lambda \in I_j} f_r(\lambda), \qquad \overline{m}_j(r) = \max_{\lambda \in I_j} f_r(\lambda).$$

In addition, the function $f_r(\lambda)$ is a continuous function of r. Therefore, for $\lambda \in I_j$ the following relationship holds uniformly with respect to λ

$$\lim_{r \to 0} f_r(\lambda) = f(\lambda), \tag{5.18}$$

from which the existence of $r_0 > 0$ follows, such that -- with allowance for (5.16) -- for $r \leqslant r_0$ we have /56

$$\underline{m}_j(r) < 0, \qquad \overline{m}_j(r) > 0. \tag{5.19}$$

Inequalities (5.19) indicate that the points $\overline{\lambda}_1 = \overline{\lambda_1(r)}, \ldots, \overline{\lambda}_j = \overline{\lambda}_j(r), \ldots,$ $\overline{\lambda}_n = \overline{\lambda}_n(r)$ are found in the segments $I_1, \ldots, I_n$, such that

$$f_r(\overline{\lambda}_j) = 0 \text{ for } j = 1, \ldots, n. \tag{5.20}$$

The values λ_j, $j = 1, \ldots, n$, are real and different, and moreover they are the eigen values of the matrix $\overline{A} = A + r_0 \frac{\overline{A} - A}{r_0} = A + rA_1,$ which is the arbitrary matrix of $\mathfrak{A}_{r_0}$. This fact proves the validity of the lemma.

We should also point out that for any matrix $\overline{A}$ of $\mathfrak{A}_{r_0}$ we may always select the matrix $\overline{U}$ which reduces $\overline{A}$ to the Jordan form $\overline{J}$:

$$\overline{A} = \overline{U}\overline{J}\overline{U}^{-1} \tag{5.21}$$

so that the following inequality is fulfilled

$$|\overline{U}||\overline{U}^{-1}| \leqslant \overline{c}, \tag{5.22}$$

in which $\overline{c}$ is a constant, which is general for all matrices of $\mathfrak{A}_{r_0}$.

Let us now study the equation

$$i(k\omega)Y_k^{(j)} + Y_k^{(j)}A_{j-1} = A_{j-1}Y_k^{(j)} + P_k^{(j-1)} \quad (|k| \neq 0), \tag{5.23}$$

where A_{j-1} is the arbitrary matrix of $\mathfrak{A}_{r_0}$.

In view of the statements presented in § 2, equation (5.23) always has a solution, such that

$$|Y_k^{(j)}| \leqslant \bar{c}_0 |k|^{d_1} |P_k^{(j-1)}|, \tag{5.24}$$

where $\bar{c}_0 = \bar{c}^2 c_2 \varepsilon^{-\frac{d_1}{d}}$, c_2, d_1 are constants determined according to (2.23).

Let us assume that the matrix $P^{(j)}(\phi)$ is periodic with respect to ϕ with the period 2π, and analytic in the region

$$|\mathrm{Im}\,\varphi| \leqslant \varrho_j \; (j = 0, 1, \ldots) \tag{5.25}$$

and satisfies the inequality

$$|P^{(j)}(\varphi)| \leqslant M_j, \tag{5.26}$$

where ϱ_j, M_j are the constants which are connected by the relationships (3.9). Taking the inequality (5.24) into account, we may then write

$$|Y_k^{(j)}| \leqslant \bar{c}_0 |k|^{d_1} M_{j-1} e^{-\varrho_{j-1}|k|}, \tag{5.27}$$

as soon as $P_k^{(j-1)} = (2\pi)^{-m} \int_0^{2\pi} \ldots \int_0^{2\pi} P^{(j-1)}(\varphi) e^{-i(k\varphi)} d\varphi_1 \ldots d\varphi_m$.

It follows from (5.27) that the matrix

$$U^{(j)}(\varphi) = \sum_{|k| \neq 0} Y_k^{(j)} e^{i(k\varphi)} \tag{5.28}$$

is the solution of equation

$$\frac{\partial U^{(j)}}{\partial \varphi} \omega + U^{(j)} A_{j-1} = A_{j-1} U^{(j)} + P^{(j-1)} - \overline{P^{(j-1)}}, \tag{5.30}$$

which is analytic in the region

$$|\mathrm{Im}\,\varphi| \leqslant \varrho_j \tag{5.29}$$

and which satisfies the inequality /57

$$|U^{(j)}(\varphi)| \leqslant \frac{M_{j-1}^{\varkappa-1}}{4n}. \tag{5.31}$$

Let us now turn to the direct proof of theorem 3. For this purpose, let us make the following substitution of the variables in system (5.6)

$$x = [E + U^{(1)}(\varphi)]\, x^{(1)}, \qquad (5.32)$$

setting $j = 1$, $A_0 = A$, $P^{(0)} = P$ in (5.23).

The substitution (5.32) reduces system (5.6) to the following form

$$\frac{dx^{(1)}}{dt} = A_1 x^{(1)} + P^{(1)}(\varphi)\, x^{(1)}, \quad \frac{d\varphi}{dt} = \omega, \qquad (5.33)$$

where we have

$$A_1 = A + \overline{P(\varphi)}, \quad P^{(1)} = (E + U^{(1)})^{-1}[PU^{(1)} - U^{(1)}\bar{P}]. \qquad (5.34)$$

The matrix $P^{(1)}(\phi)$ is analytic in the region

$$|\operatorname{Im}\varphi| \leqslant \varrho_1$$

and satisfies the inequality

$$|P^{(1)}(\varphi)| \leqslant \sum_{i=0}^{\infty} \left(\frac{M_i^{\varkappa-1}}{4n} \right)^i 2M_0 \cdot \frac{M_0^{\varkappa-1}}{4n} \leqslant M_0^{\varkappa} = M_1. \qquad (5.35)$$

Let us assume that M_0 is so small that

$$\sum_{i=0}^{\infty} M_i < r_0, \qquad (5.36)$$

where r_0 is determined by the lemma. When (5.36) is fulfilled, the matrix A_1 belongs to the set $\mathfrak{A}_{r_0}$, since

$$|A_1 - A| = |\overline{P(\varphi)}| \leqslant M_0 < r_0. \qquad (5.37)$$

We may therefore make the following substitution of the variables in system (5.33)

$$x^{(1)} = [E + U^{(2)}(\varphi)]\, x^{(2)} \qquad (5.38)$$

and may turn to the system

$$\frac{dx^{(2)}}{dt} = A_2 x^{(2)} + P^{(2)}(\varphi)\, x^{(2)}, \quad \frac{d\varphi}{dt} = \omega, \qquad (5.39)$$

where we have

$$A_2 = A_1 + \overline{P^{(1)}(\varphi)} = A + \overline{P(\varphi)} + \overline{P^{(1)}(\varphi)},$$
$$P^{(2)}(\varphi) = (E + U^{(2)})^{-1}(P^{(1)}U^{(2)} - U^{(2)}\overline{P^{(1)}}); \qquad (5.40)$$

Thus, $P^{(2)}(\phi)$ is the analytic function of ϕ in the region

$$|\mathrm{Im}\,\varphi| \leqslant \varrho_2, \tag{5.41}$$

which satisfies the inequality

$$|P^{(2)}(\varphi)| \leqslant M_2, \tag{5.42}$$

and A_2 is the matrix of $\mathfrak{A}_{r_0}$.

Continuing the process given above, we may state that the substitution of the variables /58

$$x = \prod_{j=1}^{s} [E + U^{(j)}(\varphi)]\, x^{(s)} = \Phi^{(s)}(\varphi)\, x^{(s)} \tag{5.43}$$

reduces the initial system of equations (5.6) to the following form

$$\frac{dx^{(s)}}{dt} = A_s x^{(s)} + P^{(s)}(\varphi)\, x^{(s)}, \quad \frac{d\varphi}{dt} = \omega, \tag{5.44}$$

where

$$A_s = \sum_{i=0}^{s-1} \overline{P^{(i)}(\varphi)} + A,$$

$$P^{(s)} = (E + U^{(s)})\,(P^{(s-1)}U^{(s)} - U^{(s)}\overline{P^{(s-1)}}); \tag{5.45}$$

Thus, $\Phi^{(s)}$ and $P^{(s)}$ are the analytic functions of ϕ in the region

$$|\mathrm{Im}\,\varphi| \leqslant \varrho_s \tag{5.46}$$

which satisfy the inequalities

$$|\Phi^{(s)}(\varphi)| \leqslant \left|\prod_{j=1}^{s}\left(E + \frac{M_{j-1}^{\varkappa-1}}{4n} I\right)\right| \leqslant n^2 \prod_{j=1}^{s}\left(1 + \frac{M_{j-1}^{\varkappa-1}}{4}\right), \tag{5.47}$$

when A_s is the matrix of $\mathfrak{A}_{r_0}$.

It follows from (5.47) that

$$\Phi^{(s)}(\varphi) \to \Phi(\varphi) \text{ for } s \to \infty \tag{5.48}$$

uniformly with respect to ϕ for the region

$$|\mathrm{Im}\,\varphi| \leqslant \frac{\varrho_0}{2}. \tag{5.49}$$

Therefore, the substitution of

$$x = \Phi(\varphi)\, y = \prod_{j=1}^{\infty} (E + U^{(j)}(\varphi))\, y \tag{5.50}$$

reduces the initial system of equations (5.6) to the form

$$\frac{dy}{dt} = A^0 y,\ \frac{d\varphi}{dt} = \omega, \tag{5.51}$$

where

$$A^0 = A + \sum_{j=0}^{\infty} \overline{P^{(j)}(\varphi)}. \tag{5.52}$$

The relationships (5.14) readily follow from (5.47), which completes the proof for the theorem.

In conclusion, we would like to point out that theorem 3 remains in force for periodic systems (ϕ is a scalar), if the condition that the eigen values of the matrix A are real and different is replaced by the condition

$$\lambda_j - \lambda_l + ik\omega \neq 0 \tag{5.53}$$

in the case of j, l = 1, 2, ..., n; k = 1, 2, ...

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